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A UNIFORMLY CONVEX HEREDITARILY INDECOMPOSABLE BANACH SPACE

BY

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ABSTRACT

We construct a uniformly convex hereditarily indecomposable Banach space, using a method similar to the one of Gowers and Maurey in [GM], and the theory of complex interpolation for a family of Banach spaces of Coifman, Cwikel, Rochberg, Sagher, and Weiss ([CCRSW1]).

0.1 INTRODUCTION. In [GM], W. T. Gowers and B. Maurey constructed the first known example of a space without an unconditional basic sequence. Their space, which we shall denote by X_{GM}, has even the stronger property of being **hereditarily indecomposable**. A Banach space X is said to be hereditarily indecomposable (or H.I.) if no infinite-dimensional subspace of X is decomposable, that is, no infinite-dimensional subspace of X can be written as a topological direct sum of two infinite-dimensional subspaces. In other words, a space X is H.I. if for any infinite-dimensional subspaces Y and Z of X, any $\epsilon > 0$, there exist vectors $y \in Y$, $z \in Z$, such that $\|y\| = \|z\| = 1$ and $\|y - z\| \leq \epsilon$. Later on, W. T. Gowers showed the following **dichotomy theorem**: every Banach space X contains a subspace that is either spanned by an unconditional basis, or is hereditarily indecomposable ([G]). Because of this theorem, it is of particular interest to know about general properties of H.I. spaces.

The space constructed by Gowers and Maurey is reflexive, however it is not uniformly convex. In this article, we provide an example of a complex uniformly convex hereditarily indecomposable space, using a Gowers–Maurey type

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construction; when quoting a lemma from [GM], we will denote it by the letters GM. Before giving some explanation of the proof, we need to fix some simple notation common to all spaces with a Gowers–Maurey type construction.

Notation: In the following, by **space** (resp. **subspace**), we shall always mean infinite-dimensional Banach space (resp. closed subspace).

Let c_{00} be the space of sequences of scalars all but finitely many of which are zero. Let $e_1, e_2, \ldots$ be its unit vector basis. If $E \subset \mathbb{N}$, then we shall also use the letter E for the projection from c_{00} to c_{00} defined by $E(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i \in E} a_i e_i$. If $E, F \subset \mathbb{N}$, then we write $E < F$ to mean that $\sup E < \inf F$. An **interval** of integers is a subset of $\mathbb{N}$ of the form $\{a, a+1, \ldots, b\}$ for some $a, b \in \mathbb{N}$. For N in $\mathbb{N}$, E_N denotes the interval $\{1, \ldots, N\}$. The **range** of a vector x in c_{00}, written $\mathrm{ran}(x)$, is the smallest interval E such that $Ex = x$. We shall write $x < y$ to mean $\mathrm{ran}(x) < \mathrm{ran}(y)$; notice that this is only defined on c_{00}. If $x_1 < \cdots < x_n$ we shall say that $x_1, \ldots, x_n$ are **successive**.

Let $\mathcal{X}$ be the class of Banach sequence spaces such that $(e_i)_{i=1}^{\infty}$ is a normalized bimonotone basis. Notice that for $p \geq 1$, l_p is in $\mathcal{X}$. We denote by $B(l_p)$ the unit ball of $l_p \cap c_{00}$.

By a **block basis** in a space $X \in \mathcal{X}$ we mean a sequence $x_1, x_2, \ldots$ of successive non-zero vectors in X (such a sequence must be a basic sequence) and by a **block subspace** of a space $X \in \mathcal{X}$ we mean a subspace generated by a block basis.

Let f be the function $\log_2(x+1)$. If $X \in \mathcal{X}$, and all successive vectors $x_1, \ldots, x_n$ in X satisfy the inequality $f(n)^{-1} \sum_{i=1}^{n} \|x_i\| \leq \|\sum_{i=1}^{n} x_i\|$, then we say that X satisfies an **f-lower estimate**. We denote by $\mathcal{X}(f)$ the set of Banach spaces in $\mathcal{X}$ satisfying an f-lower estimate.

A function h: $[1, +\infty) \to [1, +\infty)$ belongs to the Schlumprecht class $\mathcal{F}$ of functions if it satisfies the following five conditions:

(i) $h(1) = 1$ and $h(x) < x$ for every $x > 1$;

(ii) h is strictly increasing and tends to infinity;

(iii) $\lim_{x \to \infty} x^{-q} h(x) = 0$ for every $q > 0$;

(iv) the function $x/h(x)$ is concave and non-decreasing;

(v) $h(xy) \leq h(x)h(y)$ for every $x, y \geq 1$.

We notice that f and $\sqrt{f}$ belong to $\mathcal{F}$.

Given X in $\mathcal{X}$, given g in $\mathcal{F}$, a functional x^* in X^* is an (M, g)-form if $\|x^*\|^* \leq 1$ and $x^* = \sum_{j=1}^{M} x_j^*$ for some sequence $x_1^* < \cdots < x_M^*$ of successive functionals such that $\|x_j^*\|^* \leq g(M)^{-1}$ for each j. Notice that if $X \in \mathcal{X}(f)$, then every (M, f)-form is in the unit ball of X^*.

Before passing to some notation specific to this article, let us give an idea of our construction. The space X_{GM} satisfies an f-lower estimate; such spaces are easily shown to contain l_1^n's uniformly, so are not uniformly convex. This proximity to l_1 is a very important feature of Gowers–Maurey space. Indeed a Gowers–Maurey construction yields lower estimates for expressions of the form $\|\sum_{i=1}^n x_i\|$, but not upper estimates. For a space close to l_1, this is not a problem because an accurate upper estimate for such expressions is automatically given by the triangle inequality. But for a space close to l_p for some $p > 1$ as the one we would like to obtain, we need a better upper estimate than the triangle inequality.

Classical complex interpolation is known to produce such upper estimates, and by a result of Cwikel and Reisner ([CR]), the interpolation space of X_0 and X_1 is uniformly convex whenever X_0 or X_1 is uniformly convex; so it seems a good idea to define our space as such an interpolation space. However, it is not clear that the classical interpolation space of, say, X_{GM} and l_2 is H.I.. Gowers–Maurey construction seems to be too subtle to pass to the interpolate.

To solve this problem, a solution is to make a Gowers–Maurey type construction directly in a space X, and to make sure at the same time that X appears as the interpolate of a space close to X_{GM} and of a uniformly convex space. To do this, it will be necessary to use the more complicated notion of interpolation of a family of Banach spaces (we will call this interpolation the generalized interpolation as opposed to the classical one). Generalized interpolation has properties similar to the classical one with respect to uniform convexity and duality. Before giving its definition (all details and proofs can be found in [CCRSW1], [CCRSW2]), let us explain why we need such a notion.

The dual unit ball of Gowers–Maurey space can be thought of as a ball with a lot of spikes (corresponding to the set of special vectors of divergent normalization). The existence of these spikes is the feature that allows the H.I. property to appear. To build a H.I. space by interpolation, one would need the spikes to be preserved by interpolation; in other words, one would need sufficiently many analytic functions taking values in any part of the spikes. It is not clear that this is possible using classical interpolation; but it is made possible with generalized interpolation, because the spaces on the border can vary. More precisely, we will build spaces X_t for $t \in \mathbb{R}$ such that, given a spike in the unit ball of, say, X_0^*, if F is an analytic function such that $F(0)$ is at the point of the spike, then for every t in $\mathbb{R}$, $F(it)$ defines the point of a spike in the unit ball of X_t^*. The existence of the necessary analytic functions describing spikes in the unit balls of X_t^* follows automatically.

0.2 INTERPOLATION FOR A FAMILY OF BANACH SPACES. *Notation:* If F is an analytic function with values in c_{00}, then there exists an interval E such that for all $x \in S$, $\text{ran}(F(x)) \subset E$. We define the **range** of F to be the smallest of these intervals E. We also define successive analytic functions with values in c_{00} in an obvious way.

Let $q > 1$ in $\mathbb{R}$, q' such that $1/q + 1/q' = 1$. Let $\theta \in \,]0,1[$, and p be the number defined by $1/p = 1 - \theta + \theta/q$.

Let S be the strip $\{z \in \mathbb{C}\colon \text{Re}(z) \in [0,1]\}$, δS its boundary, S_0 the line $\{z \in \mathbb{C}\colon \text{Re}(z) = 0\}$, S_1 the line $\{z \in \mathbb{C}\colon \text{Re}(z) = 1\}$. Let μ be the Poisson probability measure associated to the point θ for the strip S, that is $d\mu(z) = P_\theta(z)dz$ where P_θ is defined by

$$P_\theta(j + it) = (e^{-\pi t} \sin \pi\theta)/(\sin^2 \pi\theta + (\cos \pi\theta - e^{ij\pi - \pi t})^2),$$

for $j = 0, 1$, $t \in \mathbb{R}$. We have $\mu(S_0) = 1 - \theta$. Let μ_0 be the probability measure on $\mathbb{R}$ defined by $\mu_0(A) = \mu(iA)/(1-\theta)$, μ_1 be the probability measure on $\mathbb{R}$ defined by $\mu_1(A) = \mu(1 + iA)/\theta$. Let $\mathcal{A}_S$ be the set of analytic functions F on S, with values in c_{00}, which are L_1 on δS for $d\mu$ and which satisfy the Poisson integral representation $F(z_0) = \int_{\delta S} F(z) dP_{z_0}(z)$ on S (this is well defined since such functions have finite ranges). If F is analytic and bounded on S, then $F \in \mathcal{A}_S$.

THEOREM 1 (Coifman, Cwikel, Rochberg, Sagher, Weiss): *Let $\|.\|_z$ for z in S be a family of norms on $\mathbb{C}^N$, such that $z \mapsto \|x\|_z$ is measurable for all x in $\mathbb{C}^N$. Assume these norms are equivalent with log-integrable constants (that is, there exist log-integrable functions c and C on S such that for all $z \in S$, all $x \in \mathbb{C}^N$, $c(z)|x| \le \|x\|_z \le C(z)|x|$, where $|.|$ is the euclidean norm). Then for any $r \ge 1$, the following formula defines a norm on $\mathbb{C}^N$:*

$$\|x\|^r = \inf_{F \in \mathcal{A}_S^N,\ F(\theta) = x} \left(\int_{z \in \delta S} \|F(z)\|_z^r d\mu(z) \right),$$

where $\mathcal{A}_S^N$ denotes the image of the canonical projection from $\mathcal{A}_S$ into the space of functions from S to $\mathbb{C}^N$. Furthermore, this norm does not depend on r. It defines a normed space called the θ-interpolation space of the family of norms (or equivalently, of the family of associated normed spaces).

We now generalize this theorem to the infinite-dimensional case as follows.

Definition 1: Let $\{X_z, z \in \delta S\}$ be a family of Banach spaces in $\mathcal{X}$, equipped with norms $\|.\|_z$, such that for all x in c_{00}, the function $z \mapsto \|x\|_z$ is measurable. For every z, the norm $\|.\|_z$ is greater than the l_∞-norm and smaller than the l_1-norm, so over vectors of finite range N, the norms $\|.\|_z$ are equivalent with log-integrable constants. Let X_z^N be $E_N X_z$, let X^N be the θ-interpolation space of the family X_z^N. The interpolation space of the family at θ is defined as completion $(\bigcup_{N\in\mathbb{N}} X^N)$.

Remarks: Notice that we only use finite range vectors in this construction.

This definition coincides with the usual definition of complex interpolation for two Banach spaces Y_0 and Y_1, if we take $X_z = Y_i$ when $z \in S_i$, for $i = 0, 1$.

This article is divided into three sections. In the first one, we define and study a class $\mathcal{X}_\theta$ of uniformly convex generalized interpolation spaces; in the second one, we construct a particular space X in this class. In the last section, we show that X is hereditarily indecomposable.

1. A class of uniformly convex Banach spaces

All the results in this section use only properties that generalized interpolation shares with classical interpolation. We first specify the less general context in which we will use generalized interpolation.

1.1 Definition and notation. Let $\{X_t, t \in \mathbb{R}\}$ be a family of spaces in $\mathcal{X}$, equipped with norms $\|.\|_t$, such that for all t in $\mathbb{R}$, X_t satisfies a f-lower estimate and for all x in c_{00}, the function $t \mapsto \|x\|_t$ is measurable. By Theorem 1, we may define the θ-interpolation space X of the family defined on δS as X_t if $z = it$, l_q if $z = 1 + it$. We shall sometimes use for $z \in \delta S$ the notation $\|.\|_z$, to mean $\|.\|_t$ if $z = it$, and $\|.\|_{q'}$ if $z = 1 + it$. There will be no ambiguity from the context. We shall similarly use the notation $\|.\|_z^*$. The notation X_t^N stands for $E_N X_t$, and X_t^{N*} for $E_N X_t^*$. Also, unless specified otherwise, the measure of a subset of $\mathbb{R}$ will be its measure for μ_0.

Let $\mathcal{X}_\theta$ be the class of spaces X obtained in the previous way. In this section, we shall show some general results about elements of $\mathcal{X}_\theta$.

1.2 PROPERTIES OF $\mathcal{X}_\theta$. Let X be in $\mathcal{X}_\theta$ and let x in X have finite range. Let $\mathcal{A}_\theta(x)$ be the set of functions in $\mathcal{A}_S$ that take the value x at the point θ. Given θ, it is the set of **interpolation functions for** x. By definition, $\|x\|$ is characterized by the following equality for any value of $r \geq 1$:

$$\|x\|^r = \inf_{F \in \mathcal{A}_\theta(x)} \left(\int_{z \in \delta S} \|F(z)\|_z^r d\mu(z) \right).$$

The following theorem is a useful result of [CCRSW1].

THEOREM 2: *If x has finite range, then there is an interpolation function F for x, that we shall call* **minimal for** x, *with* $\operatorname{ran}(F) = \operatorname{ran}(x)$ *and such that*

$$\|F(it)\|_t = \|x\| \text{ a.e.} \quad \text{and} \quad \|F(1+it)\|_q = \|x\| \text{ a.e.}$$

LEMMA 1: *The following formula is also true: for any $r \geq 1$,*

$$\|x\|^r = \inf_{F \in \mathcal{A}_\theta(x)} \left(\int_{\mathbb{R}} \|F(it)\|_t^r d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q^r d\mu_1(t) \right)^{\theta}.$$

Proof: First notice that for any F in $\mathcal{A}_\theta(x)$, by a convexity inequality, the argument in the second infimum is smaller than

$$(1-\theta) \left(\int_{\mathbb{R}} \|F(it)\|_t^r d\mu_0(t) \right) + \theta \left(\int_{\mathbb{R}} \|F(1+it)\|_q^r d\mu_1(t) \right)$$

equal to $\int_{z \in \delta S} \|F(z)\|_z^r d\mu(z)$, so that the second infimum is smaller than the first one.

Now, given $u \in \mathbb{R}$, the map G_u defined on $\mathcal{A}_\theta(x)$ by $G_u(F)(z) = F(z)e^{u(z-\theta)}$ is a bijection on $\mathcal{A}_\theta(x)$. Furthermore, for any u, the expressions

$$\left(\int_{\mathbb{R}} \|(G_u(F)(it)\|_t^r d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \|G_u(F)(1+it)\|_q^r d\mu_1(t) \right)^{\theta}$$

and

$$\left(\int_{\mathbb{R}} \|F(it)\|_t^r d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q^r d\mu_1(t) \right)^{\theta}$$

are equal. If we choose a proper u (namely such that the two quantities $\int_{\mathbb{R}} \|(G_u(F)(it)\|_t^r d\mu_0(t)$ and $\int_{\mathbb{R}} \|G_u(F)(1+it)\|_q^r d\mu_1(t)$ are equal), this is also equal to $\int_{z \in \delta S} \|G_u(F)(z)\|_z^r d\mu(z)$. Consequently, the two infima are actually equal. ∎

PROPOSITION 1: *For all successive vectors $x_1 < \cdots < x_n$ in X,*

$$\frac{1}{f(n)^{1-\theta}}\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{n} x_i\right\| \leq \left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p}.$$

Proof: It is enough to prove this in the interpolation space X^N defined above, written in short $((X_t^N), l_q^N)_\theta$, for any $N \geq 1$.

First inequality: The unit ball of X_t^N is stable under sums of the form $\sum_{j=1}^n \lambda_j y_j$, where the y_j are successive in the unit ball of X_t^N and $\sum_{j=1}^n |\lambda_j| = 1$.

The unit ball of l_q^N is stable under sums of the form $\sum_{j=1}^n \mu_j z_j$, where the z_j are successive in the unit ball of l_q^N and $\sum_{j=1}^n |\mu_j|^q = 1$.

Consequently, the unit ball of X^N is stable under successive sums of the form $\sum_{j=1}^n \lambda_j^{1-\theta}\mu_j^\theta x_j$, where the x_j are in the unit ball of X^N and λ_j and μ_j satisfy the above conditions. Indeed, for every x_j in the unit ball of X^N, let F_j be minimal for x_j; the function F defined by $F(z) = \sum_{j=1}^n \lambda_j^{1-z}\mu_j^z F_j(z)$ is then in $\mathcal{A}_S$ and bounded by 1 a.e. on δS, so by definition, $\|F(\theta)\| \leq 1$, that is, $\sum_{j=1}^n \lambda_j^{1-\theta}\mu_j^\theta x_j$ is in the unit ball of X^N.

Now consider any successive vectors x_j in X^N, and apply this stability property to $x_j/\|x_j\|$ and $\lambda_j = \mu_j^q = \|x_j\|^p/\sum_{i=1}^n \|x_i\|^p$. Using the equality $1-\theta+\theta/q = 1/p$, one finally gets

$$\left\|\sum_{j=1}^{n} x_i\right\| \leq \left(\sum_{j=1}^{n}\|x_i\|^p\right)^{1/p}.$$

This inequality will be called the **upper p-estimate for X**.

Second inequality: According to [CCRSW1], the duality property is true in finite dimension, that is $X^{N*} = ((X_t^{N*}), l_q^{N*})_\theta$. As X_t satisfies a lower f-estimate, so does X_t^N; the dual version of this is that the unit ball of X_t^{N*} is stable under sums of the form $(1/f(n))\sum_{j=1}^n y_j^*$, where the y_j^* are successive. As $l_q^{N*} = l_{q'}^N$, we know that its unit ball is stable under successive sums of the form $\sum_{j=1}^n \mu_j z_j^*$, where $\sum_{j=1}^n |\mu_j|^{q'} = 1$. Letting $\lambda_j = 1/f(n)$ for each j, and using the same proof as above, we get that the unit ball of X^{N*} is stable under successive sums of the form $(1/f(n)^{1-\theta})\sum_{j=1}^n \mu_j^\theta x_j^*$.

Now let x_j be successive vectors in X^N; for $j = 1, \ldots, n$, let x_j^* be successive dual unit vectors such that x_j^* norms x_j (we recall that the basis is bimonotone in every X_t, so it is bimonotone in X). We get that $(1/f(n)^{1-\theta})\sum_{j=1}^n \mu_j^\theta \|x_j\| \leq$

$\|\sum_{j=1}^n x_j\|$. Choosing $\mu_j^{q'} = \|x_j\|^p / \sum_{i=1}^n \|x_i\|^p$ and using the equality $\theta/q' = 1 - 1/p$ gives the desired inequality

$$\frac{1}{f(n)^{1-\theta}} \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq \left\| \sum_{i=1}^n x_i \right\|.$$

This inequality will be called the **lower estimate for** X. ∎

Remark: Gowers–Maurey space is close to l_1, in the sense that for successive vectors, the triangular inequality is, up to a logarithmic term, an equality. As the interpolation space of l_1 and l_q is l_p, one expects the space X to be close to l_p; the above inequalities show in what sense this is true.

PROPOSITION 2: *The dual space X^* of X is also the interpolation space — in the sense of Definition 1 — of the family defined on δS as X_t^* if $z = it$ and $l_{q'}$ if $z = 1 + it$.*

In other words, the duality property of usual complex interpolation is still true in our extension.

Proof: First notice that this interpolation space is well defined, because the family $\{(X_t^*)_{t \in \mathbb{R}}, l_{q'}\}$ satisfies the conditions of Definition 1. We recall that a basis $(x_n)_{n=1}^\infty$ of a Banach space is **shrinking** if for every continuous linear functional x^* and every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that the norm of x^* restricted to the span of $x_n, x_{n+1}, \dots$ is at most ϵ. The basis $e_1, e_2, \dots$ is a shrinking basis for X. Indeed, suppose it is not; then we can find $\epsilon > 0$, a norm-1 functional $x^* \in X^*$, and a sequence of successive normalized blocks $x_1, x_2, \dots$ such that $x^*(x_n) \geq \epsilon$ for every n. Then, using the upper p-estimate, we get $n\epsilon \leq x^*(\sum_{i=1}^n x_i) \leq \|\sum_{i=1}^n x_i\| \leq n^{1/p}$, a contradiction if we choose n big enough.

This implies that given x^* in X^*, $\|x^*\|_{X^*} = \lim_{N \to +\infty} \|E_N x^*\|_{X^{N*}}$. But this means that $X^* = \text{completion}\,(\bigcup_{n \in \mathbb{N}} X^{N*})$; according to [CCRSW1], X^{N*} is also the interpolation space $((X_t^N)^*, l_{q'}^N)_\theta$; as it is easy to show that $(X_t^N)^* = (X_t^*)^N$, we get the desired dual property. ∎

PROPOSITION 3: *The space X is uniformly convex.*

Proof: This result is similar to the result of Cwikel and Reisner in the case of the classical complex interpolation of two spaces ([CR]). It is enough to prove that any vectors x and y in the unit ball of X^N satisfy the relation

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\|x-y\|)$$

where δ is strictly positive on $]0,+\infty[$ and does not depend on N.

Suppose $q \geq 2$. Then for any vectors a and b in the unit ball of l_q^N,

$$\left\|\frac{a+b}{2}\right\|_q^q \leq 1 - \left\|\frac{a-b}{2}\right\|_q^q$$

(the proof of this Clarkson's inequality can be found for example in [B]). Now let x and y be in the unit ball of X^N, let F (resp. G) be a minimal interpolation function for x (resp. y) as in Theorem 2. Let us apply Lemma 1 with $r = q$:

$$\left\|\frac{x+y}{2}\right\|^q \leq \left(\int_{\mathbb{R}} \left\|\frac{F+G}{2}(it)\right\|_t^q d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \left\|\frac{F+G}{2}(1+it)\right\|_q^q d\mu_1(t)\right)^{\theta}.$$

The first integral is smaller than 1, so that

$$\left\|\frac{x+y}{2}\right\|^q \leq \left(\int_{\mathbb{R}} \left\|\frac{F+G}{2}(1+it)\right\|_q^q d\mu_1(t)\right)^{\theta}.$$

Similarly,

$$\left\|\frac{x-y}{2}\right\|^q \leq \left(\int_{\mathbb{R}} \left\|\frac{F-G}{2}(1+it)\right\|_q^q d\mu_1(t)\right)^{\theta}.$$

Adding these two estimates together and using Clarkson's estimate we get

$$\left\|\frac{x+y}{2}\right\|^{q/\theta} + \left\|\frac{x-y}{2}\right\|^{q/\theta} \leq 1.$$

If $q < 2$, the estimate is slightly different: there is a constant c_q such that for any vectors a and b in the unit ball of l_q^N,

$$\left\|\frac{a+b}{2}\right\|_q \leq 1 - c_q\|a-b\|_q^2.$$

Applying the same method as above, we obtain

$$\left\|\frac{x+y}{2}\right\|^{1/\theta} + c_q\|x-y\|^{2/\theta} \leq 1.$$

In both the cases $q \geq 2$ and $q < 2$, the inequalities above are uniform convexity inequalities. ∎

We are now going to define particular vectors in a space X in $\mathcal{X}_\theta$. In [GM], two kinds of vectors are considered: l_1^{n+}-vectors and R.I.S. vectors; l_1^{n+}-vectors are maximal, and R.I.S. vectors minimal with respect to the inequalities

$$f(n)^{-1}\sum_{i=1}^{n}\|x_i\| \leq \|\sum_{i=1}^{n} x_i\| \leq \sum_{i=1}^{n}\|x_i\|.$$

Here we consider modifications of these definitions for vectors in a space in $\mathcal{X}_\theta$, taking into account that such a space is close to l_p for $p > 1$: these new vectors are called l_p^{n+}-vectors (Definition 2) and again R.I.S. vectors (Definition 3). They have the same properties as their equivalents in Gowers–Maurey space: minimality (Lemma 10), maximality (Definition 2), existence (Lemma 2). There are also properties that link the two versions of each type of vector (Lemma 3, Lemma 4).

1.3 l_{p+}^n-AVERAGES.

Definition 2: Let n be a non-zero integer, C a real number.

Let X be in $\mathcal{X}$. An **l_{1+}^n-average in X with constant C** is a normalized vector $x \in X$ such that $x = \sum_{i=1}^n x_i$, where $x_1 < \cdots < x_n$ are successive vectors and each x_i verifies $\|x_i\| \leq Cn^{-1}$.

Let X be in $\mathcal{X}_\theta$. An **l_{p+}^n-average in X with constant C** is a normalized vector $x \in X$ such that $x = \sum_{i=1}^n x_i$, where $x_1 < \cdots < x_n$ are successive vectors and each x_i verifies $\|x_i\| \leq Cn^{-1/p}$.

An **l_{1+}^n (resp. l_{p+}^n) -vector** is a non-zero multiple of an l_{1+}^n (resp. l_{p+}^n)-average.

LEMMA 2: *Let X be in $\mathcal{X}_\theta$. For every $n \geq 1$, every $C > 1$, every block subspace Y of X contains an l_{p+}^n-average with constant C.*

Proof: The proof is the same as in Lemma 3 of [GM]. Suppose the result is false for some Y. Let k be an integer such that $k\log C > (1-\theta)\log f(n^k)$, let $N = n^k$, let $x_1 < \cdots < x_N$ be any sequence of successive norm-1 vectors in Y, and let $x = \sum_{i=1}^N x_i$. For every $0 \leq i \leq k$ and every $1 \leq j \leq n^{k-i}$, let $x(i,j) = \sum_{t=(j-1)n^i+1}^{jn^i} x_t$. Thus $x(0,j) = x_j, x(k,1) = x$, and, for $1 \leq i \leq k$, each $x(i,j)$ is a sum of n successive $x(i-1,j)$'s. By our assumption, no $x(i,j)$ is an l_{p+}^n-vector with constant C. It follows easily by induction that $\|x(i,j)\| \leq C^{-i}n^{i/p}$

and, in particular, that $\|x\| \leq C^{-k}n^{k/p} = C^{-k}N^{1/p}$. However, it follows from the lower estimate in X that $\|x\| \geq N^{1/p}f(N)^{-(1-\theta)}$. This is a contradiction, by choice of k. ∎

LEMMA 3: *Let X be in $\mathcal{X}_\theta$. Let $0 < \epsilon < 1/4$. Let $\theta = 1/2$. Let x be an l_{p+}^n-average in X with constant $1+\epsilon$. Then there exists an interpolation function F for x with* $\text{ran}(F) = \text{ran}(x)$, *bounded almost everywhere by $1+\epsilon$, such that except on a set of measure at most $2\sqrt{\epsilon}$, $F(it)$ is an l_{1+}^n-vector in X_t, of norm 1 up to $\sqrt{\epsilon}$, with constant $1+4\sqrt{\epsilon}$.*

Such a function is called **ϵ-representative**, or **representative**, since we shall always consider l_{p+}^n-averages associated to given values of ϵ.

Proof: The vector x can be written $\sum_{j=1}^n x_j$, where $x_1 < \cdots < x_n$ are successive vectors and each x_j verifies $\|x_j\| \leq (1+\epsilon)n^{-1/p}$. Let F_j' be a minimal interpolation function for x_j, let F_j be defined by $F_j(z) = n^{-1/p'+z/q'}F_j'(z)$ and let $F = \sum_{j=1}^n F_j$. We show that F is representative for x.

Notice that $F(\theta) = x$, so F is an interpolation function for x, and

$$1 = \|x\| \leq \left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t)\right)^{\theta}.$$

By choice of F, F is bounded by $1+\epsilon$ a.e. on δS, so both integrals are smaller than $1+\epsilon$. As a consequence, $\int_{t\in\mathbb{R}} \|F(it)\|_t d\mu_0(t) \geq (1+\epsilon)^{-\theta/(1-\theta)} \geq 1-\epsilon$ (recall that $\theta = 1/2$). As for every t, $\|F(it)\|_t \leq 1+\epsilon$, by a Bienaymé–Tchebitschev estimation, we get that on a set of measure at least $1-2\sqrt{\epsilon}$, $\|F(it)\|_t \geq 1-\sqrt{\epsilon}$.

So on that set, $F(it)$ is of norm 1 up to $\sqrt{\epsilon}$. For each j,

$$\|F_j(it)\|_t = n^{-1/p'}\|x_j\| \leq (1+\epsilon)/n;$$

so that $F(it)$ is an l_{1+}^n-vector in X_t with constant $(1+\epsilon)/(1-\sqrt{\epsilon}) \leq 1+4\sqrt{\epsilon}$. ∎

1.4 RAPIDLY INCREASING SEQUENCES. From now on, we assume that $\theta = 1/2$.

Definition 3: Let N be a non-zero integer. Let $0 < \epsilon \leq 1$.

Let X be in $\mathcal{X}_\theta$. A sequence $x_1 < \cdots < x_N$ in X is a **Rapidly Increasing Sequence of l_{p+}^n-averages, or R.I.S., of length N with constant $1+\epsilon$** if x_k is an $l_{p+}^{n_k}$-average with constant $1+\epsilon/n_k$ for each k, $n_1 \geq 4M_f(N/\epsilon)/\epsilon f'(1)$, and $\epsilon/2\ f(n_k)^{1/2} \geq |\text{ran}(x_{k-1})|$ for $k = 2, \ldots, N$.

Here $f'(1)$ is the right derivative of f at 1 and M_f is defined on $[1,\infty)$ by $M_f(x) = f^{-1}(36x^2)$.

In spaces X_t, we shall use R.I.S. in the Gowers–Maurey sense, that is, sequences of $l_{1+}^{n_k}$-averages with constant $1+\epsilon$ with the same increasing condition as above.

We shall call both versions "R.I.S." without ambiguity; when using both versions at the same time, we shall denote by $x_1 < \cdots < x_n$ a R.I.S. in a space X_t, and by $X_1 < \cdots < X_n$ a R.I.S. in an interpolation space X.

A R.I.S.-vector is a non-zero multiple of the sum of a R.I.S. The following proposition links the two versions of R.I.S.

LEMMA 4: *Let X be in $\mathcal{X}_\theta$. Let $0 < \epsilon \leq 1/16$. Let $X_1 < \cdots < X_n$ be a R.I.S. in X with constant $1+\epsilon$. For each k, let F_k be representative for X_k; then $F = F_1 + \cdots + F_n$ is an interpolation function for $\sum_{k=1}^n X_k$, and except on a set of measure at most $4\sqrt{\epsilon}/f(n)$, $F(it)$ is up to $2\sqrt{\epsilon}$ the sum of a R.I.S. in X_t with constant $1+4\sqrt{\epsilon}$.*

Proof: It is clear that F is an interpolation function for $\sum_{k=1}^n X_k$. According to Lemma 3, for each k, $F_k(it)$ is 'close' to an $l_{1+}^{n_k}$-average, except on a set of measure at most $2\sqrt{\epsilon/n_k}$. The union over k of these sets is of measure at most $\sum_{k=1}^n 2\sqrt{\epsilon/n_k} \leq 4\sqrt{\epsilon/n_1} \leq 4\sqrt{\epsilon}/f(n)$ (this is a consequence of the increasing condition and the lower bound for n_1 in the definition of the R.I.S.).

Now let t be in this union. For every k, let $|F|_k(it)$ denote the normalization of $F_k(it)$; $|F|_k(it)$ is an $l_{1+}^{n_k}$-average with constant $1+4\sqrt{\epsilon/n_k}$. The sequence $|F|_1(it) < \cdots < |F|_n(it)$ is a R.I.S. in X_t, with constant $\sup_k(1+4\sqrt{\epsilon/n_k}) \leq 1+4\sqrt{\epsilon}$ (because $1+4\sqrt{\epsilon} > 1+\epsilon$, the increasing condition is indeed verified).

It remains to show that $F(it)$ and the sum of the $|F|_k(it)$ are equal up to $2\sqrt{\epsilon}$; and indeed $\|F(it) - \sum_{k=1}^n |F|_k(it)\|_t \leq \sum_{k=1}^n |1 - \|F_k(it)\||_t \leq \sum_{k=1}^n \sqrt{\epsilon/n_k} \leq 2\sqrt{\epsilon}$, so that the proof is complete. ∎

Special sequences: We are now going to make a Gowers–Maurey type construction, defining special sequences. We are going to define special sequences of **dual interpolation functions**. The idea behind this is the following: for every t in $\mathbb{R}$, the set of values in t of these special sequences can be considered as a set of special sequences for X_t. Thus, by a Gowers–Maurey construction, we obtain for every t a space X_t that "looks like" Gowers–Maurey space. But, because of the way we build special sequences, the special property of the X_t is somehow uniform on t, which allows one to carry Gowers–Maurey type estimations from the spaces X_t into the interpolate. The construction of the space, which is rather technical, is developed in the next section.

2. Construction of a space X in $\mathcal{X}_\theta$

2.1 CONSTRUCTION OF SPACES X_t. Let $J = \{j_1, j_2, \ldots\}$, where $(j_n)_{n\in\mathbb{N}}$ is a sequence of integers such that $f(j_1) > 256$ and $\log\log\log j_n > 4(j_{n-1})^2$ for $n > 1$. Let $K = \{j_1, j_3, j_5, \ldots\}$ and $L = \{j_2, j_4, j_6, \ldots\}$. Let $\{L_m, m \in \mathbb{N}^*\}$ be a partition of L with every L_m infinite. For $r \in [1, +\infty]$, let $B(l_r)$ denote the unit ball of $l_r \cap c_{00}$. For $N \geq 1$ and $z \in \mathbb{C}$, let $f(N,z) = f(N)^{1-z}N^{z/q'}$ and $g(N,z) = \sqrt{f(N)}^{1-z}N^{z/q'}$.

Definition 4: Given a subset D of $\mathcal{A}_S$, for every $N > 0$, a **N-Schlumprecht sum in** D is a function of the form $f(N,z)^{-1}\sum_{i=1}^N F_i$, where the F_i are successive in D. A **Schlumprecht sum in** D is a N-Schlumprecht sum in D for some $N > 0$. For every $N > 0$, let $\Sigma_N(D)$ be the set of N-Schlumprecht sums in D, and let $\Sigma(D)$ be the set of Schlumprecht sums. If D is countable, given an injection τ from $\bigcup_{m\in\mathbb{N}}\Sigma(D)^m$ to $\mathbb{N}$, and an integer k, a **special sequence in** D**, for** τ**, with length** k, is a sequence $G_1 < \cdots < G_k$ of successive functions satisfying $G_j \in \Sigma_{n_j}(D)$ for $j = 1, \ldots, k$, $n_1 = j_{2k}$ and $n_j = \tau(G_1, \ldots, G_{j-1})$ for $j = 2, \ldots, k$. A **special function in** D**, for** τ**, with length** k is a function of the form $g(k,z)^{-1}\sum_{j=1}^k G_j$, where $G_1 < \cdots < G_k$ is a special sequence.

Here, it does not seem possible to define the set of special functions before defining the spaces X_t as in [GM], so we build them at the same time by induction. More precisely, we build by induction a set $D(t)$ for any t, whose closure will be the dual unit ball of X_t, a set $\mathcal{D}$, meant to be almost equal to the set of dual interpolation functions for the interpolation of $(X_t)_{t\in\mathbb{R}}$ and (l_q), and a countable set Δ dense in $\mathcal{D}$, meant to be the set of special interpolation functions.

STEP 1: For every t in $\mathbb{R}$, let $D_1(t) = B(l_1)$. Let $\mathcal{D}_1$ be the set of functions in $\mathcal{A}_S$ with values in $D_1(t)$ for almost every it and in $B(l_{q'})$ almost everywhere on S_1. Let Δ_1 be a countable set of functions in $\mathcal{A}_S$, dense in $\mathcal{D}_1$ for the L_1-norm (namely $\|F\| = \int_{z\in\delta S}\|F(z)\|_1 d\mu(z)$). For this first step, we may assume that all functions in Δ_1 are continuous. Let σ_1 be an injection from $\bigcup_{m\in\mathbb{N}}(\Delta_1)^m$ to L_1, the first subset of L in the partition $\{L_m, m \in \mathbb{N}^*\}$. Let $S_0^1 = \mathbb{R}$.

STEP n: We are given a set of sequences $D_{n-1}(t)$ for every t in $\mathbb{R}$, a set $\mathcal{D}_{n-1}$ of functions in $\mathcal{A}_S$, a countable set Δ_{n-1} of functions in $\mathcal{A}_S$ defined everywhere on

S_0, a subset S_0^{n-1} of $\mathbb{R}$ of measure 1, and an injection σ_{n-1} from $\bigcup_{m\in\mathbb{N}}(\Delta_{n-1})^m$ to $L_1 \cup \cdots \cup L_{2n-3}$. Here we may not assume that all functions in Δ_{n-1} are continuous; that is why we introduce a set S_0^{n-1} of 'significant' values of the functions in Δ_{n-1}, which we may assume to be of measure 1 because Δ_{n-1} is countable.

Then let $\Delta'_n = \Sigma(\Delta_{n-1}) \cup \{EF, E \textit{ interval}, F \in \Delta_{n-1}\}$. Notice that Δ'_n is a countable set containing Δ_{n-1}. Let τ_{n-1} be an injection from $\bigcup_{m\in\mathbb{N}}({\Delta'}_n^m \setminus \Delta_{n-1}^m)$ to L_{2n-2}.

Let $\mathcal{S}_{n-1}$ be the set of special functions in Δ_{n-1}, for τ_{n-1}, with length in K. For every t in $\mathbb{R}$, let $D_n^S(t)$ be the sets of sequences of the form $f(N)^{-1}\sum_{i=1}^N x_i$ where the x_i are successive in $D_{n-1}(t)$ (that is, coming from the usual Schlumprecht operation); let $D_n^I(t)$ be the set of sequences Ex where E is an interval and x is in $D_{n-1}(t)$; if $t \in S_0^{n-1}$, let $D_n^s(t)$ be the set of sequences of the form $G(it)$ where $G \in \mathcal{S}_{n-1}$; otherwise, let $D_n^s(t) = \emptyset$ ($D_n^s(t)$ is the set of sequences coming from a special operation similar to the one in [GM], only if t belongs to the 'significant' set of values S_0^{n-1}). Let $D'_n(t) = D_n^S(t) \cup D_n^I(t) \cup D_n^s(t)$ and let $D_n(t) = \operatorname{conv}(\bigcup_{|\lambda|=1} \lambda D'_n(t))$. Let $\mathcal{D}_n$ be the set of functions in $\mathcal{A}_S$ with values in $D_n(t)$ for almost every it and in $B(l_{q'})$ almost everywhere on S_1.

We complete Δ'_n in a countable subset Δ_n of $\mathcal{D}_n$, dense in $\mathcal{D}_n$ for the L_1-norm. As Δ_n is countable, there is a subset $S_0^n \subset S_0^{n-1}$ of $\mathbb{R}$ of measure 1 such that $F(it)$ is indeed in $D_n(t)$ for all F in Δ_n and for all t in S_0^n. With an injection τ'_{n-1}, from $\bigcup_{m\in\mathbb{N}}(\Delta_n^m \setminus {\Delta'}_n^m)$ to L_{2n-1}, we obtain an injection σ_n, from $\bigcup_{m\in\mathbb{N}}(\Delta_n)^m$ to $L_1 \cup \cdots \cup L_{2n-1}$.

Definition of X_t: It is easy to verify that the sequences $D_n(t)$ for every t in $\mathbb{R}$, $\mathcal{D}_n$ and Δ_n are increasing, that the sequence S_0^n is decreasing and that for every n, σ_n coincides with σ_{n-1} on its set of definition.

We then define $D_t = \bigcup_{n\in\mathbb{N}} D_n(t)$ for every t in $\mathbb{R}$, $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$, $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$, $S_0^\infty = \bigcap_{n\in\mathbb{N}} S_0^n$ and σ the injection from $\bigcup_{m\in\mathbb{N}} \Delta^m$ to L whose restrictions are the σ_n.

Finally, for every t in $\mathbb{R}$, we define on c_{00} a norm $\|.\|_t$ by

$$\forall x \in c_{00}, \|x\|_t = \sup_{y\in D(t)} |\langle x, y\rangle|.$$

We let X_t be the completion of c_{00} for this norm.

2.2 Properties of $\mathcal{D}$ and Δ.

Proposition 4:

(a) *For every t in $\mathbb{R}$, $B(l_1) \subset D(t) \subset B(l_\infty)$.*
(b) *The set Δ is countable, dense in $\mathcal{D}$, stable under interval projections and Schlumprecht sums in Δ.*
(c) *For every t in $\mathbb{R}$, the set $D(t)$ is convex, balanced, stable under interval projections, and sums of the form $f(N)^{-1}\sum_{i=1}^N x_i$, with $x_i \in D(t)$ and $x_1 < \cdots < x_N$.*
(d) *The set $\mathcal{D}$ is convex, balanced, stable under interval projections, Schlumprecht sums in $\mathcal{D}$, and under taking special functions in Δ for σ with length in K.*

Proof: (a) The left inclusion is a consequence of the facts that $D_1(t) = B(l_1)$ and that $n \to D_n(t)$ is increasing; for the right inclusion, notice that by induction, $D_n(t) \subset B(l_\infty)$ for every n.

(b) The set Δ is countable as a countable union of countable sets; it is dense in $\mathcal{D}$ because for every n, Δ_n is dense in $\mathcal{D}_n$; the stability property under interval projections and Schlumprecht sums is ensured because for every n, Δ_n contains Δ'_n, the set of projections and sums from Δ_{n-1}.

(c) The set $D(t)$ is convex as an increasing union of convex sets; the stability properties are ensured by the definition of $D'_n(t)$ and $D_n(t)$ from $D_{n-1}(t)$.

(d) The set $\mathcal{D}_n$ is the set of functions with values in the convex, balanced, and interval projection stable sets $D_n(t)$ and $B(l_{q'})$ on δS, so that it is convex, balanced and stable under interval projections; and so is $\mathcal{D}$.

To show the Schlumprecht stability property, it is enough, given successive functions $F_1 < \cdots < F_N$ in $\mathcal{D}_{n-1}$, to show that $F = f(N,z)^{-1}\Sigma_{j=1}^N F_j$ is in $\mathcal{D}_n$. For each j, $F_j(it)$ is in $D_{n-1}(t)$ almost everywhere. The set of $t \in \mathbb{R}$ such that this happens for every j is still of measure 1. On this set, $F(it) = (f(N)N^{-1/q'})^{it}(1/f(N))\sum_{j=1}^N F_j(it)$ is in $D_n(t)$, by the definition of $D_n(t)$. In the same way, almost everywhere on S_1, $F_j(1+it)$ is in $B(l_{q'})$ for every j, so that $F(1+it)$ is in $B(l_{q'})$ too. By definition, this means that F is in $\mathcal{D}_n$.

To show the special property, first notice that a special function G in Δ is a special function in Δ_n for some n in $\mathbb{N}$. It follows that $G(it) \in D_{n+1}(t)$ for every t in S_0^n, that is, almost everywhere; furthermore, $G(1+it)$ is in $B(l_{q'})$ almost everywhere; so G is in $\mathcal{D}_{n+1}$. ∎

We now need a technical lemma, which explains what we meant by 'almost' in the explanation of the definition of $\mathcal{D}$ just after Definition 4.

LEMMA 5: *Let $\mathcal{S}$ be the set of functions in $\mathcal{A}_S$ with values in $D(t)$ for almost every it and in $B(l_{q'})$ almost everywhere on S_1. Then $\mathcal{D}$ is dense in $\mathcal{S}$ for the L_1-norm.*

Proof: We first recall the Havin lemma from [P] in a rougher version. Furthermore, we state it on S instead of on the unit disk of $\mathbb{C}$ (the two versions are equivalent using a conformal mapping).

LEMMA: *For every $\epsilon' > 0$, there exists $\delta > 0$ such that for every subset e of δS with $\mu(e) \leq \delta$, there exists g_e in $H^\infty(S)$ with $|g_e| \leq 1$ a.e. on δS, $\sup_{z\in e}|g_e(z)| \leq \epsilon'$, and $\int_{\delta S}|g_e(z)-1|d\mu(z) \leq \epsilon'$.*

Let F be in $\mathcal{S}$, $0<\epsilon<1$. Let N be such that $\text{ran}(F) \subset E_N$. Now let δ be associated to $\epsilon' = \epsilon/N$ in the above lemma. The sequence of sets $(T_n)_{n\in\mathbb{N}}$ defined by $T_n = \{t\colon F(it) \in D_n(t)\}$ is increasing and its union is of measure 1 for μ_0, so there exists n such that T_n is of measure at least $1-\delta$. For μ, $\delta S \setminus iT_n$ is of measure at most $\delta(1-\theta) \leq \delta$. Let H be the function $g_{\delta S \setminus iT_n}$. Let $\tilde{F} = H.F$. The function $\tilde{F}$ is in $\mathcal{A}_S$. Furthermore, $\tilde{F}(1+it)$ is in $B(l_{q'})$ almost everywhere on S_1, $\tilde{F}(it)$ is in $D_n(t)$ almost everywhere on T_n; this last assertion is also true on $S_0 \setminus T_n$, because almost everywhere on this set, $\tilde{F}(it)$ is in $1/N\ D(t)$ and because, for functions of range at most E_N, we have, with a slight abuse of notation, the following inclusions: $1/N\ D(t) \subset 1/N\ B(l_\infty) \subset B(l_1) \subset D_n(t)$. This proves that $\tilde{F}$ is in $\mathcal{D}_n$.

It remains to show that $F-\tilde{F}$ is small in the L_1-norm, and indeed:

$$\int_{\delta S}\|(F-\tilde{F})(z)\|_1 d\mu(z) \leq N\int_{\delta S}|H(z)-1|d\mu(z) \leq \epsilon. \qquad \blacksquare$$

2.3 DEFINITION OF X. For every x in c_{00}, the function $t \mapsto \|x\|_t$ is measurable. To see it, it is enough to prove that the restriction of the function to S_0^∞ is measurable. We prove this by induction on $|\text{ran}(x)|$. Remember that $\|x\|_t = \sup_{y\in D(t)}|\langle x,y\rangle|$. Now let y be in $D(t)$; then y is a convex combination of elements of the following form: either elements of $B(l_1)$, or values in it of projections of special functions, or n-Schlumprecht sums with $n>1$. The supremum $\|x\|_t$ is certainly attained in an element of one of these three kinds, so that

$$\|x\|_t = \|x\|_\infty \bigvee \sup_{G\text{ special},E}|\langle x, EG(it)\rangle| \bigvee \sup_{n\geq 2, \mathcal{E}_1<\cdots<\mathcal{E}_n}\frac{1}{f(n)}\sum_{j=1}^n\|\mathcal{E}_j x\|_t.$$

In the last supremum, $\mathcal{E}_1 < \cdots < \mathcal{E}_n$ are any successive intervals. We may restrict this supremum to the finite set of intervals $\mathcal{E}_j$ contained in $\operatorname{ran}(x)$ and different from it. The first supremum is over a countable set. Finally, $t \mapsto \|x\|_t$ is the supremum of a countable family of measurable functions by the induction hypothesis, so it is a measurable function.

Furthermore, it follows from the stability property of $D(t)$ that for every t in $\mathbb{R}$, X_t satisfies an f-lower estimate. So all the conditions are satisfied for the definition of a space in $\mathcal{X}_\theta$.

Definition 5: We denote by X the space $((X_t), l_q)_\theta$ of $\mathcal{X}_\theta$.

LEMMA 6: *Let $F^* \in \mathcal{D}$. Then $F^*(\theta)$ is in the unit ball of X^*.*

Proof: First notice that if we restrict them to finite range vectors, it is a consequence of their convexity and of the definition of $\|.\|_t$ that the unit ball of X_t^* and $\overline{D(t)}$ coincide. Now, given F^* in $\mathcal{D}$, it is of finite range. For almost every t, $F^*(it) \in D(t)$, so that by the previous remark, $\|F^*(it)\|_t^* \leq 1$. Furthermore, $\|F^*(1+it)\|_{q'} \leq 1$, so by Proposition 2, $\|F^*(\theta)\|^* \leq 1$. ∎

We now state two lemmas that are an easy modification of Lemma GM7 of [GM] for the first one and a mixture of Lemmas GM8 and GM9 of [GM] for the second one.

LEMMA 7: *Let $f, g \in \mathcal{F}$ with $g \geq \sqrt{f}$, let $X \in \mathcal{X}$ satisfy a lower f-estimate, let $0 < \epsilon \leq 1$, let $x_1 < \cdots < x_N$ be a R.I.S. in X for f with constant $1+\epsilon$, and let $x = \sum_{i=1}^N x_i$. Suppose that*

$$\|Ex\| \leq 1 \vee \sup\{|x^*(Ex)|: M \geq 2, x^* \text{ is an } (M,g)\text{-form}\}$$

for every interval E. Then $\|x\| \leq (1+2\epsilon)Ng(N)^{-1}$.

LEMMA 8: *Let $K_0 \subset K$. Let ϕ: $[1,\infty) \mapsto [1,\infty)$ be defined by*

$$\phi(x) = \sqrt{f(x)} \quad \text{if } x \in K_0, \qquad \phi(x) = f(x) \quad \text{otherwise.}$$

Then there is a function g: $[1,\infty) \mapsto [1,\infty)$ such that: $g \in \mathcal{F}$, $\sqrt{f} \leq g \leq \phi \leq f$, and if $N \in J \setminus K_0$, then $g = f$ on the interval $[\log N, \exp N]$.

Now we prove that the property of minimality of the R.I.S., an important feature of Gowers–Maurey space, is also true in every X_t (Lemma 9), and in X (Lemma 10).

LEMMA 9: *Let $t \in \mathbb{R}$. Let $N \in L$, let $n \in [\log N, \exp N]$, let $0 < \epsilon \leq 1$, and let $x_1 < \cdots < x_n$ be a R.I.S. in X_t with constant $1+\epsilon$. Then*

$$\left\|\sum_{i=1}^{n} x_i\right\|_t \leq (1+2\epsilon)n/f(n).$$

Proof: The space $X_t \in \mathcal{X}$ satisfies a lower f-estimate.

Let x be the sum of the R.I.S. $x_1 < \cdots < x_n$. Let E be any interval. Let ϕ be the function defined in Lemma 8 in the case $K_0 = K$ and g associated to ϕ by the same lemma. The vector Ex is normed by a functional x^* in $D(t)$. If x^* is in $D_1(t)$, then $\|Ex\|_t = |x^*(Ex)| \leq 1$. Else there exists $m \geq 2$ such that x^* is in $D_m(t) \setminus D_{m-1}(t)$; then, by definition of $D_m(t)$, x^* is a convex combination of (M, f)-forms with $M \geq 2$ and $(M, \sqrt{f})$-forms with $M \in K$; since $g \leq \phi$, it follows that x^* is a convex combination of (M, g)-forms with $M \geq 2$; so Ex can be normed by an (M, g)-form. Consequently,

$$\|Ex\|_t \leq 1 \vee \sup\{|x^*(Ex)|: M \geq 2,\ x^* \text{ is an } (M,g)\text{-form}\}$$

Since $g \in \mathcal{F}$ and $g \geq \sqrt{f}$, all the hypotheses of Lemma 7 are satisfied. It follows that $\|\sum_{i=1}^{n} x_i\|_t \leq (1+2\epsilon)n/g(n)$. By Lemma 8, $g(n) = f(n)$, which proves our statement. ∎

LEMMA 10: *We recall that X denotes the space given by Definition 5. Let $N \in L$, let $n \in [\log N, \exp N]$, let $0 < \epsilon \leq 1/16$, and let $X_1 < \cdots < X_n$ be a R.I.S. in X with constant $1+\epsilon$. Then*

$$\left\|\sum_{i=1}^{n} X_i\right\| \leq (1+10\sqrt{\epsilon})n^{1/p}/f(n)^{1-\theta}.$$

Proof: Let F_k be representative for X_k, and $F = F_1 + \cdots + F_n$. We know that F is an interpolation function for $X_1 + \cdots + X_n$ so

$$\left\|\sum_{i=1}^{n} X_i\right\| \leq \left(\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t)\right)^{1-\theta} \left(\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t)\right)^{\theta}.$$

For the second integral, the following estimate holds:

$$\int_{\mathbb{R}} \|F(1+it)\|_q d\mu_1(t) \leq (1+\epsilon)n^{1/q}.$$

According to Lemma 4, there is a set A of measure at most $4\sqrt{\epsilon}/f(n)$ such that on $\mathbb{R} \setminus A$, $F(it)$ is up to $2\sqrt{\epsilon}$ the sum x_t of a R.I.S. in X_t. On $\mathbb{R} \setminus A$, $\|F(it)\|_t \leq$

$\|x_t\|_t + 2\sqrt{\epsilon}$; furthermore, x_t is a R.I.S. in X_t with constant $1 + 4\sqrt{\epsilon}$, so that by Lemma 9, $\|x_t\|_t \leq (1 + 8\sqrt{\epsilon})n/f(n)$. On A, we have only $\|F(it)\|_t \leq (1 + \epsilon)n$.

Gathering these estimates, we get

$$\int_{\mathbb{R}} \|F(it)\|_t d\mu_0(t) \leq \left[(1 + 8\sqrt{\epsilon})\frac{n}{f(n)} + 2\sqrt{\epsilon}\right] + \frac{4\sqrt{\epsilon}}{f(n)}(1 + \epsilon)n \leq (1 + 15\sqrt{\epsilon})\frac{n}{f(n)}.$$

Going back to the R.I.S. $X_1 < \cdots < X_n$, we have

$$\left\|\sum_{i=1}^{n} X_i\right\| \leq (1 + 15\sqrt{\epsilon})^{1-\theta}(1 + \epsilon)^{\theta}\frac{n^{1-\theta+\theta/q}}{f(n)^{1-\theta}} \leq (1 + 10\sqrt{\epsilon})\frac{n^{1/p}}{f(n)^{1-\theta}}. \qquad ∎$$

The following lemma is similar to Lemma GM11 in [GM].

Lemma 11: *Let $t \in \mathbb{R}$. Let $N \in L$, let $0 < \epsilon < 1/4$, let $M = N^{\epsilon}$ and let $x_1 < \cdots < x_N$ be a R.I.S. in X_t with constant $1 + \epsilon$. Then $\sum_{i=1}^{N} x_i$ is an l_{1+}^{M}-vector in X_t with constant $1 + 4\epsilon$.*

Proof: Let $m = N/M$, let $x = \sum_{i=1}^{N} x_i$ and for $1 \leq j \leq M$ let $y_j = \sum_{i=(j-1)m+1}^{jm} x_i$. Then each y_j is the sum of a R.I.S. of length m with constant $(1 + \epsilon)$. By Lemma 9 we have $\|y_j\|_t \leq (1 + 2\epsilon)mf(m)^{-1}$ for every j while $\|\sum_{j=1}^{m} y_j\|_t = \|x\| \geq Nf(N)^{-1}$. It follows that x is an l_{1+}^{M}-vector in X_t with constant at most $(1+2\epsilon)f(N)/f(m)$. But $m = N^{1-\epsilon}$ so $f(N)/f(m) \leq (1-\epsilon)^{-1}$. The result follows. ∎

We now pass to the crucial lemma of this construction. It is the equivalent of Lemma GM12 of [GM], but involves vectors in X_t for some t and special dual functions. Its proof is essentially identical to the one of Lemma GM12.

Lemma 12: *Let $\epsilon_0 = 1/10$. Let $k \in K$ and $F_1^*, \ldots, F_k^*$ be a special sequence of length k, with $F_i^* \in \Sigma_{M_i}(\Delta)$. Let $t \in S_0^{\infty}$. Let $x_1 < \cdots < x_k$ a sequence of successive vectors in X_t, where every x_i is a normalized R.I.S.-vector of length M_i and constant $1 + \epsilon_0/4$. Suppose $\mathrm{ran}(F_i^*) \subset \mathrm{ran}(x_i)$ for $i = 1, \ldots, k$, and $1/2\ \epsilon_0 f(M_i^{\epsilon_0/4})^{1/2} \geq |\mathrm{ran}(x_{i-1})|$ for $i = 2, \ldots, k$.*

If for every interval E, $|(\sum_{i=1}^{k} F_i^*(t))(\sum_{i=1}^{k} Ex_i)| \leq 4$, then

$$\left\|\sum_{i=1}^{k} x_i\right\|_t \leq (1 + 2\epsilon_0)k/f(k).$$

Proof: First we recall two lemmas from [GM].

LEMMA GM4: *Let $M, N \in \mathbb{N}$ and $C \geq 1$, let $X \in \mathcal{X}$, let $x \in X$ be an l_{1+}^N-vector with constant C and let $\mathcal{E}_1 < \cdots < \mathcal{E}_M$ be a sequence of intervals. Then $\sum_{j=1}^M \|\mathcal{E}_j x\| \leq C(1+2M/N)\|x\|$.*

LEMMA GM5: *Let $f, g \in \mathcal{F}$ with $g \geq f^{1/2}$ and let $X \in \mathcal{X}$ satisfy a lower f-estimate. Let $0 < \epsilon \leq 1$, let $x_1 < \cdots < x_N$ be a R.I.S. in X with constant $1+\epsilon$ and let $x = \sum_{i=1}^N x_i$. Let $M \geq M_f(N/\epsilon)$, let x^* be an (M, g)-form and let E be any interval. Then $|x^*(Ex)| \leq 1 + 2\epsilon$.*

According to Lemma 11, each x_i is an $l_{1+}^{N_i}$-average with constant $1+\epsilon_0$, where $N_i = M_i^{\epsilon_0/4}$. The increasing condition and the lower bound for M_1 ensure that $x_1 < \cdots < x_k$ is a R.I.S. in X_t of length k with constant $1+\epsilon_0$.

To prove this Lemma we shall apply Lemma 7. First, we show that if $G_1^*, \ldots, G_k^*$ is any special sequence in Δ of length k and E is any interval, then $|z^*(Ex)| < 1$, where z^* is the $(k, \sqrt{f})$-form $f(k)^{-1/2} \sum_{i=1}^k z_i^*$ with $z_j^* = G_j^*(it)$, and $x = \sum_{i=1}^k x_i$.

Indeed, let s be maximal such that $G_s^* = F_s^*$ or zero if no such s exists. Suppose now $i \neq j$ or one of i, j is greater than $s+1$. Then since σ is an injection, we can find $L_1 \neq L_2 \in L$ such that z_i^* is an (L_1, f)-form and x_j is the normalized sum of a R.I.S. of length L_2 and also an $l_{1+}^{L_2'}$-average with constant $1+\epsilon_0$, where $L_2' = L_2^{\epsilon_0/4}$. We can now use Lemmas GM4 and GM5 to show that $|z_i^*(Ex_j)| < k^{-2}$.

If $L_1 < L_2$, it follows from the lacunarity of L that $L_1 < L_2'$. We know that $L_1 \geq j_{2k}$ since L_1 appears in a special sequence of length k. Lemma GM4 thus gives $|z_i^*(Ex_j)| = |(Ez_i^*)(x_j)| \leq 3(1+\epsilon_0)/f(L_1)$. The conclusion in this case now follows from the fact that $f(l) \geq 4k^2$ when $l \geq j_{2k}$.

If $L_2 < L_1$, we apply Lemma GM5 in X_t with $\epsilon = 1$ to the non-normalized sum x_j' of the R.I.S. the normalized sum of which is x_j. The definition of L gives us that $M_f(L_2) < L_1$, so Lemma GM5 gives $|z_i^*(Ex_j')| \leq 3$. It follows from the lower f-estimate in X_t that $\|x_j'\| \geq L_2/f(L_2)$. The conclusion now follows because $l \geq j_{2k}$ implies that $f(l)/l \leq 1/4k^2$.

Now choose an interval E' such that

$$\left| \left(\sum_{i=1}^s z_i^* \right) (Ex) \right| = \left| \left(\sum_{i=1}^k F_i^*(it) \right) (E'x) \right| \leq 4.$$

It follows that

$$\left|\left(\sum_{i=1}^{k} z_i^*\right)(Ex)\right| \leq 4 + |z_{s+1}^*(x_{s+1})| + k^2.k^{-2} \leq 6.$$

We finally obtain that $|z^*(Ex)| \leq 6f(k)^{-1/2} < 1$ as claimed.

Now let ϕ' be the function

$$\phi'(x) = \sqrt{f(x)} \quad \text{if } x \in K, \quad x \neq k, \quad \phi'(x) = f(x) \quad \text{otherwise.}$$

Let g' be the function obtained from ϕ' by Lemma 8 in the case $K_0 = K \setminus \{k\}$; we know that $g'(l) = f(l)$ for every $l \in L \cup \{k\}$.

It follows from what we have just shown about special sequences of length k that for every interval E,

$$\|Ex\|_t \leq 1 \vee \sup\{|x^*(Ex)|\colon M \geq 2, x^* \text{ is an } (M, g')\text{-form}\}.$$

Since x is the sum of a R.I.S., Lemma 7 implies that $\|x\|_t \leq (1+2\epsilon_0)kg'(k)^{-1} = (1+2\epsilon_0)k/f(k)$. ∎

3. X is hereditarily indecomposable

Let Y and Z be two infinite-dimensional subspaces of X. We want to show that their sum is not a topological sum. We may assume that $Y \cap Z = \{0\}$. Let $\delta > 0$. We shall build two vectors $y \in Y$ and $z \in Z$ such that $\delta\|y+z\| > \|y-z\|$. The existence of such vectors for any $\delta > 0$ easily implies that the canonical projection from $Y+Z$ onto Y is not continuous.

Let $\epsilon_0 = 1/10$. Let $k \in K$ be an integer such that $1/4 < \epsilon_0\, k^{1/p}/f(k)^{1-\theta}$ and $2/\sqrt{f(k)}^{1-\theta} < \delta$, and let $\epsilon > 0$ be such that $\sqrt{\epsilon} \leq \epsilon_0/4kf(k)$. We may assume that both Y and Z are spanned by block bases. We now build a sequence $(x_j)_{j=1}^k$ in X by iteration.

3.1 Proposition: *Let Y, Z, k be as above. There exist successive sequences (F_j) of interpolation functions, $(\mathcal{F}_j^*)$ of dual interpolation functions, (x_j) of vectors in X, (x_j^*) of linear forms in X^*, and a sequence (M_j) of integers such that:*

x_j is a norm 1 R.I.S. vector with length M_j and constant $1+\epsilon$, belonging to Y when j is odd, to Z otherwise;

x_j^* is of norm at most 1;
$\mathcal{F}_1^*, \ldots, \mathcal{F}_k^*$ is a special sequence of length k, with $\mathcal{F}_j^* \in \Sigma_{M_j}(\Delta)$;
$x_j = F_j(\theta)$ up to $10\sqrt{\epsilon}$ and $x_j^* = \mathcal{F}_j^*(\theta)$;
for $j = 1, \ldots, k$, $\mathrm{ran}(\mathcal{F}_j^*) \subset \mathrm{ran}(x_j^*) \subset \mathrm{ran}(x_j) = \mathrm{ran}(F_j)$;
$M_1 = j_{2k}$ and for $j = 2, \ldots, k$, $1/2\epsilon_0 f(M_j^{\epsilon_0/4})^{1/2} \geq |\,\mathrm{ran}(F_{j-1})|$;
for every j, $\langle \mathcal{F}_j^*(\theta), F_j(\theta)\rangle = 1$ up to ϵ;
for every j, except on $\mathcal{J}_j$ of measure at most $2\sqrt{\epsilon}$, $\langle \mathcal{F}_j^*(it), F_j(it)\rangle = 1$ up to $2\sqrt{\epsilon}$;
for every j, except on $\mathcal{J}_j'$ of measure at most $4\sqrt{\epsilon}/f(M_j)$, $F_j(it)$ is up to $10\sqrt{\epsilon}$ the normalized sum of a R.I.S. with constant $1 + 4\sqrt{\epsilon} \leq 1 + \epsilon_0/4$.

Proof: We build these sequences by induction. We first explain how we define the first elements of the sequences.

FIRST STEP: By Lemma 2, Y and Z contain, for every $N \in \mathbb{N}$, an l_{p+}^N-average with constant $1 + \epsilon$. Let $x_1 \in Y$ be a R.I.S.-vector of norm 1, constant $1 + \epsilon$ and length $M_1 = j_{2k}$; we have $M_1^{\epsilon_0/4} = N_1 \geq 4M_f(k/\epsilon_0)/\epsilon_0 f'(1)$. Let $x_{11} < \cdots < x_{1M_1}$ be the R.I.S. whose normalized sum is x_1: there exists λ_1 such that $\lambda_1 x_1 = x_{11} + \cdots + x_{1M_1}$. Applying the lower estimate in X and Lemma 10, we get

$$M_1^{1/p}/f(M_1)^{1-\theta} \leq \|\lambda_1 x_1\| \leq (1 + 10\sqrt{\epsilon})M_1^{1/p}/f(M_1)^{1-\theta},$$

so that $\lambda_1 = M_1^{1/p}/f(M_1)^{1-\theta}$ up to the multiplicative factor $1 + 10\sqrt{\epsilon}$.

Now for $m = 1, \ldots, M_1$, we associate to x_{1m}:

a representative function F_{1m} for x_{1m};
a vector x_{1m}^* in X^* that norms x_{1m} and with $\mathrm{ran}(x_{1m}^*) \subset \mathrm{ran}(x_{1m})$;
a minimal interpolation function F_{1m}^* for x_{1m}^*; it exists because of Proposition 2 and because, as x_{1m}^* is of finite range, Theorem 2 applies.

The function F_{1m}^* is in $\overline{\mathcal{S}}$. Indeed, remember that if we restrict them to finite range vectors, the unit ball of X_t^* and $\overline{D(t)}$ coincide; so by the convexity of $D(t)$, for every $\nu > 0$, the function $F_{1m}^*/(1+\nu)$ takes its values in $D(t)$ for almost every it; as it takes its values in $B(l_{q'})$ a.e. on S_1, it is in $\mathcal{S}$, which ends the proof that F_{1m}^* is in $\overline{\mathcal{S}}$.

By Lemma 5, F_{1m}^* can be approached by a function $\mathcal{F}_{1m}^*$ in Δ; and because of the interval projection stability of Δ, we may assume that $\mathrm{ran}(\mathcal{F}_{1m}^*) \subset \mathrm{ran}(F_{1m}^*)$. More precisely, we suppose that $\mathcal{F}_{1m}^*$ is close to F_{1m}^* up to $\epsilon/(1+\epsilon)$ for the norm

$\int_{z\in\delta S} \|.\|_z^* d\mu(z)$. This is possible because over functions of finite range, this norm is smaller than the norm $\int_{z\in\delta S} \|.\|_1 d\mu(z)$ first introduced.

Lastly, we define two functions:

Let $\mathcal{F}_1^* = f(M_1, z)^{-1} \sum_{m=1}^{M_1} \mathcal{F}_{1m}^*$. It belongs to Δ. Let $x_1^* = \mathcal{F}_1^*(\theta)$.

Let $F_1 = f(M_1, z) M_1^{-1} \sum_{m=1}^{M_1} F_{1m}$.

ITERATION: To simplify the notation, we show how to pass from the first step to the second. It will be clear that we can repeat the same procedure at any step until k.

Let $M_2 = \sigma(\mathcal{F}_1^*) \in L$. We may assume that we chose $\mathcal{F}_1^*$ such that M_2 satisfies $1/2\ \epsilon_0 f(M_2^{\epsilon_0/4})^{1/2} \geq |\operatorname{ran}(F_1)|$. Indeed, as Δ is dense in $\overline{S}$, for every m, there are infinitely many possible choices for the functions $\mathcal{F}_1^*$, leading to infinitely many possible values for M_2, all different because σ is an injection. So we can assume that M_2 is as big as we want, in particular, that it satisfies the above inequality. Let x_2 be a R.I.S.-vector of norm 1 in Z, with constant $1+\epsilon$ and length M_2, satisfying $x_2 > x_1$. We then define x_2^*, F_2, $\mathcal{F}_2^*$ in the same way as in the first step of the iteration.

Of the proposition, only the last three points are not obvious and remain to be checked.

For every j, $\langle \mathcal{F}_j^*(\theta), F_j(\theta)\rangle = 1$ up to ϵ.

For F and F^* in $\mathcal{A}_S$, define $\langle F, F^*\rangle$ to be $\int_{z\in\delta S}\langle F(z), F^*(z)\rangle d\mu(z)$, and notice that this is equal to $\langle F(\theta), F^*(\theta)\rangle$ by analyticity. Now for every j,

$$\langle \mathcal{F}_j^*, F_j\rangle = \frac{1}{M_j}\sum_{m=1}^{M_j}\langle \mathcal{F}_{jm}^*, F_{jm}\rangle.$$

If we replace each $\mathcal{F}_{jm}^*$ by F_{jm}^*, the sum is equal to

$$\frac{1}{M_j}\sum_{m=1}^{M_j}\langle x_{jm}^*, x_{jm}\rangle = 1.$$

The error we make by doing this is $|1/M_j \sum_{m=1}^{M_j}\langle \mathcal{F}_{jm}^* - F_{jm}^*, F_{jm}\rangle|$, smaller than

$$\frac{1}{M_j}\sum_{m=1}^{M_j}\int_{z\in\delta S}\|(\mathcal{F}_{jm}^* - F_{jm}^*)(z)\|_z^* \|F_{jm}(z)\|_z d\mu(z) \leq \frac{1}{M_j}\sum_{m=1}^{M_j}\frac{\epsilon}{1+\epsilon}(1+\epsilon) \leq \epsilon$$

(we recall that as F_{jm} is representative for x_{jm}, $\|F_{jm}(z)\|_z \leq 1+\epsilon$ almost everywhere).

For every j, except on $\mathcal{J}_j$ of measure at most $2\sqrt{\epsilon}$, $\langle \mathcal{F}_j^*(it), F_j(it)\rangle = 1$ up to $2\sqrt{\epsilon}$.

Let F_j^* be the function $f(M_1, z)^{-1}\sum_{m=1}^{M_1} F_{jm}^*$. It is easy to see that

$$1 = \int_{z\in\delta S} \langle F_j^*(z), F_j(z)\rangle d\mu(z),$$

while

$$\langle F_j^*(z), F_j(z)\rangle \leq 1+\epsilon \quad \text{a.e.}$$

By a Bienaymé–Tchebitschev estimation, except on a set of measure at most $\sqrt{\epsilon}$, $\langle F_j^*(z), F_j(z)\rangle = 1$ up to $\sqrt{\epsilon}$. Furthermore, we know that

$$\int_{z\in\delta S} |\langle(\mathcal{F}_j^* - F_j^*)(z), F_j(z)\rangle| d\mu(z) \leq \epsilon$$

so that except on a set of measure at most $\sqrt{\epsilon}$, $\langle(\mathcal{F}_j^* - F_j^*)(z), F_j(z)\rangle = 0$ up to $\sqrt{\epsilon}$.

Adding these two estimates completes the proof.

For every j, except on $\mathcal{J}_j'$ of measure at most $4\sqrt{\epsilon}/f(M_j)$, $F_j(it)$ is up to $10\sqrt{\epsilon}$ the normalized sum of a R.I.S. with constant $1+4\sqrt{\epsilon} \leq 1+\epsilon_0/4$.

For each m, F_{jm} is representative for x_{jm}, so by Lemma 4, except on a set $\mathcal{J}_j'$ of measure $4\sqrt{\epsilon}/f(M_j)$, we have

$$\Big\|\sum_{m=1}^{M_j} F_{jm}(it) - x_t\Big\|_t \leq 2\sqrt{\epsilon},$$

where x_t is the sum of a R.I.S. in X_t with constant $1+4\sqrt{\epsilon}$. So

$$\left\| F_j(it) - \left(\frac{M_j^{1/q'}}{f(M_j)}\right)^{it} \frac{f(M_j)}{M_j} x_t \right\|_t \leq 2\sqrt{\epsilon}\frac{f(M_j)}{M_j} \leq 2\sqrt{\epsilon}.$$

The proof follows, because by Lemma 9,

$$M_j/f(M_j) \leq \|x_t\|_t \leq (1+8\sqrt{\epsilon})M_j/f(M_j),$$

so $f(M_j)/M_j\ x_t$ is up to $8\sqrt{\epsilon}$ a normalized R.I.S.-vector. ∎

3.2 THE SPACE X IS HEREDITARILY INDECOMPOSABLE. The proof relies on a lower estimate for $\|\sum_{j=1}^{k} x_j\|$ and on an upper estimate for $\|\sum_{j=1}^{k}(-1)^j x_j\|$. The lower estimate comes from the fact that the special sequence $\mathcal{F}_1^*,\ldots,\mathcal{F}_k^*$ norms the sequence $x_1,\ldots,x_k$. The upper estimate is the estimate of Lemma 12 in each X_t carried over to X by Lemma 1.

Estimation of $\|\sum_{j=1}^{k} x_j\|$: Let $\mathcal{G}^* = g(k,z)^{-1}\Sigma_{j=1}^{k}\mathcal{F}_j^*$. Since, by construction, the sequence $\mathcal{F}_1^*,\ldots,\mathcal{F}_k^*$ is special, $\mathcal{G}^*$ is in $\mathcal{D}$ and, by Lemma 6, $x^* = \mathcal{G}^*(\theta)$ is in the unit ball of X^*.

So $\|\sum_{j=1}^{k} F_j(\theta)\| \geq x^*(\sum_{j=1}^{k} F_j(\theta)) \geq (1-\epsilon)k^{1/p}/\sqrt{f(k)}^{1-\theta}$, and

$$\Big\|\sum_{j=1}^{k} x_j\Big\| \geq (1-\epsilon_0)k^{1/p}/\sqrt{f(k)}^{1-\theta} - 1/4 \geq (1-2\epsilon_0)k^{1/p}/\sqrt{f(k)}^{1-\theta}.$$

The $1/4$ is the error we made by replacing the x_j's by the $F_j(\theta)$'s.

Estimation of $\|\sum_{j=1}^{k}(-1)^{j-1}x_j\|$: Let $\mathcal{J}$ be the union of the $\mathcal{J}_j$'s and the $\mathcal{J}_j'$'s. The set $\mathcal{J}$ is of measure at most $6k\sqrt{\epsilon}$.

For every t in $\mathbb{R}\setminus\mathcal{J}$, for every interval E, let us evaluate

$$\left|\left(\sum_{j=1}^{k}\mathcal{F}_j^*(it)\right)\left(\sum_{j=1}^{k}(-1)^{j-1}EF_j(it)\right)\right|.$$

This is a sum of at most k scalars. Those who come from terms of range included in E are equal to $(-1)^{j-1}$ up to $2\sqrt{\epsilon}$, so that their sum is $-1, 0$ or 1 up to $2k\sqrt{\epsilon}$; two others can come from terms whose range intersects E, they are bounded in modulus by $1+10\sqrt{\epsilon}$; the others are equal to 0. So the sum is smaller than $1+2k\sqrt{\epsilon}+2(1+10\sqrt{\epsilon}) \leq 3+3k\sqrt{\epsilon}$.

For every j, $F_j(it)$ is up to $10\sqrt{\epsilon}$ a R.I.S. vector $x_j(t)$. The $(-1)^{j-1}x_j(t)$'s satisfy the hypotheses of Lemma 12: the increasing condition is satisfied, and for every interval E,

$$\left|\left(\sum_{j=1}^{k}\mathcal{F}_j^*(it)\right)\left(\sum_{j=1}^{k}(-1)^{j-1}Ex_j(t)\right)\right| \leq 3+3k\sqrt{\epsilon}+10k\sqrt{\epsilon} \leq 4.$$

It then follows from the conclusion of Lemma 12 and the relation between $F_j(t)$ and $x_j(t)$ that

$$\Big\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\Big\|_t \leq (1+2\epsilon_0)k/f(k)+10k\sqrt{\epsilon}.$$

It follows that

$$\int_{\mathbb{R}\setminus\mathcal{J}}\Big\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\Big\|_t d\mu_0(t)\le(1+2\epsilon_0)k/f(k)+10k\sqrt{\epsilon}.$$

We now want to estimate the integral of this same norm on $\mathcal{J}$. It is enough, by the triangle inequality, to evaluate $\int_{t\in\mathcal{J}}\|F_j(it)\|_t d\mu_0(t)$. If t belongs to $\mathcal{J}_j'$, by the triangle inequality, $\|F_j(it)\|_t\le(1+\epsilon)f(M_j)$, but recall that $\mathcal{J}_j'$ is of measure at most $4\sqrt{\epsilon}/f(M_j)$; else, $F_j(it)$ is up to $10\sqrt{\epsilon}$ a normalized R.I.S. vector, so that $\|F_j(it)\|_t\le 1+10\sqrt{\epsilon}$, and this on a set of measure less than $6k\sqrt{\epsilon}$. Finally,

$$\int_{\mathcal{J}}\|F_j(it)\|_t d\mu_0(t)\le 6k\sqrt{\epsilon}(1+10\sqrt{\epsilon})+\frac{4\sqrt{\epsilon}}{f(M_j)}(1+\epsilon)f(M_j)\le 7k\sqrt{\epsilon}$$

and

$$\int_{\mathcal{J}}\Big\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\Big\|_t d\mu_0(t)\le 7k^2\sqrt{\epsilon}.$$

It follows from these two estimates that

$$\int_{\mathbb{R}}\Big\|\sum_{j=1}^{k}(-1)^{j-1}F_j(it)\Big\|_t d\mu_0(t)\le(7k^2+10k)\sqrt{\epsilon}+(1+2\epsilon_0)\frac{k}{f(k)}\le(1+4\epsilon_0)\frac{k}{f(k)}.$$

Furthermore, almost everywhere on S_1,

$$\Big\|\sum_{j=1}^{k}(-1)^{j-1}F_j(1+it)\Big\|_q\le(1+\epsilon)k^{1/q},$$

so that, by Lemma 1,

$$\Big\|\sum_{j=1}^{k}(-1)^{j-1}F_j(\theta)\Big\|\le(1+3\epsilon_0)k^{1/p}/f(k)^{1-\theta},$$

and

$$\Big\|\sum_{j=1}^{k}(-1)^{j-1}x_j\Big\|\le(1+3\epsilon_0)k^{1/p}/f(k)^{1-\theta}+1/4\le(1+4\epsilon_0)k^{1/p}/f(k)^{1-\theta}.$$

Conclusion: Let $y\in Y$ be the sum of the x_j with odd indices, $z\in Z$ be the sum of the x_j with even indices. By the above estimates and by choice of k, they satisfy $\delta\|y+z\|>\|y-z\|$. As δ is arbitrary and so are Y and Z, X is hereditarily indecomposable. ∎

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